SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 11

MATH 241

19.6.14

Proof. $\deg(x^2+1)^2 = 4, \deg x^2 = 2, 4 \geq 2+2$.

Zeroes are $\pm i$, only i is on the upper half plane. However, it is not a simple pole, so we cannot use our favorite formula $\frac{P(x)}{Q'(x)}$'s evaluation. Instead, we multiply $(x-i)^2$ to $f(x) = \frac{x^2}{(x^2+1)^2}$, get $g(x) := \frac{x^2}{(x+i)^2}$. $g'(x) = \frac{2x(x+i)^2 - x^2 \cdot 2(x+i)}{(x+i)^4} = \frac{2ix}{(x+i)^3}$, so evaluate at i we get $Res(f(z), i) = g(i) = \frac{1}{4i}$. Therefore $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = 2\pi i Res(f(z), i) = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2}$. □

19.6.25

Proof. The integrand $\frac{\cos 3x}{(x^2+1)^2}$ is an even function, so $\int_0^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{1}{2} P.V. \int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{1}{2} Re(P.V. \int_{-\infty}^{\infty} \frac{e^{3ix}}{(x^2+1)^2} dx)$. Let $f(z) := \frac{1}{(z^2+1)^2}$, $\deg(z^2+1)^2 = 4$, $\deg 1 = 0$, $4 \geq 0+1$, so the condition on the degree is satisfied. The zeroes of $(z^2+1)^2$ are $\pm i$, only i is on the upper half plane. So the theorem says

$$P.V. \int_{-\infty}^{\infty} \frac{e^{3x}}{(x^2+1)^2} dx = 2\pi i Res(f(z)e^{3iz}, i)$$

i is a pole of order 2 for $f(z)e^{3iz}$, so we need to multiply $(z-i)^2$ to it. Let $g(z) := (z-i)^2 \cdot f(z)e^{3iz}$. Then $g(z) = \frac{e^{3iz}}{(z+i)^2}$. $g'(z) = \frac{3ie3iz(z+i)^2 - e^{3iz} \cdot 2(z+i)}{(z+i)^4} = \frac{(3iz-5)e^{3iz}}{(z+i)^3}$. Evaluate it at $z = i$, we get $\frac{(-8)e^{-3}}{(2i)^3}$, simplify to get $-\frac{e^{-3}}{i}$. That is the residue of $f(z)e^{3iz}$ at i.

So $P.V. \int_{-\infty}^{\infty} \frac{e^{3x}}{(x^2+1)^2} dx = 2\pi i Res(f(z)e^{3iz}, i) = 2\pi i \cdot (-\frac{e^{-3}}{i}) = -2\pi e^{-3}$. Take real part we get $P.V. \int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = -2\pi e^{-3}$, then take one half of it we get $\int_0^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{1}{2} P.V. \int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = -\pi e^{-3}$. □

19.6.30

Proof. The integrand $\frac{x \sin x}{(x^2+1)(x^2+4)}$ is an even function, so $\int_0^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)} dx = \frac{1}{2} P.V. \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)(x^2+4)} dx = \frac{1}{2} Im(P.V. \int_{-\infty}^{\infty} \frac{xe^{ix}}{(x^2+1)(x^2+4)} dx)$. Let $f(z) := \frac{z}{(z^2+1)(z^2+4)}$, $\deg(z^2+1)(z^2+4) = 4$, $\deg z = 1$, $4 \geq 1+1$, so the condition on the degree is satisfied. The zeroes of $(z^2+1)(z^2+4)$ are $\pm i, \pm 2i$, among which $i, 2i$ are on the upper half plane. So the Theorem says

$$P.V. \int_{-\infty}^{\infty} \frac{xe^{ix}}{(x^2+1)(x^2+4)} dx = 2\pi i (Res(f(z)e^{iz}, i) + Res(f(z)e^{iz}, 2i))$$

i is a simple pole of $f(z)e^{iz}$, so $Res(f(z)e^{iz}, i) = \frac{xe^{ix}}{(x^4+5x^2+4)'}|_{x=i} = \frac{ie^{-1}}{4i^3+10i} = \frac{e^{-1}}{12}$.

i is also a simple pole of $f(z)e^{iz}$, so $Res(f(z)e^{iz}, 2i) = \frac{xe^{ix}}{(x^4+5x^2+4)'}|_{x=2i} = \frac{2ie^{-2}}{4(2i)^3+10 \cdot 2i} = -\frac{e^{-2}}{6}$.

Therefore $P.V. \int_{-\infty}^{\infty} \frac{xe^x}{(x^2+1)(x^2+4)} dx = 2\pi i \cdot (\frac{e^{-1}}{12} - \frac{e^{-2}}{6}) = \frac{e^{-1}-2e^{-2}}{6}\pi i$. Take imaginary part we get $P.V. \int_{-\infty}^{\infty} \frac{xe^x}{(x^2+1)(x^2+4)} dx = \frac{e^{-1}-2e^{-2}}{6}\pi$, then take one half of it we get $\int_0^{\infty} \frac{xe^x}{(x^2+1)(x^2+4)} dx = \frac{1}{2} P.V. \int_{-\infty}^{\infty} \frac{xe^x}{(x^2+1)(x^2+4)} dx = \frac{e^{-1}-2e^{-2}}{12}$. ☐

19.6.31

Proof. $P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = Im(P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx)$. Let $f(x) := \frac{1}{x}$. Degree of the denominator is 1, of the numerator is 0, and $1 \geq 0 + 1$, so the condition on the degree is satisfied. The only pole is $x = 0$ on the real axis, and is a simple pole. So the residue is $\frac{e^{ix}}{x'}|_{x=0} = 1$, so by the formula

$$P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx = \pi i (Res(f(z)e^{iz}, 0)) = \pi i \cdot 1 = \pi i$$

So $P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$. ☐

19.6.32

Proof. $P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx = Im(P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)} dx)$. Let $f(x) := \frac{1}{x(x^2+1)}$. Degree of the denominator is 3, of the numerator is 0, and $3 \geq 0 + 1$, so the condition on the degree is satisfied.

The pole on real axis is $x = 0$, it is a simple pole, so $Res(f(z)e^{iz}, 0) = \frac{e^{ix}}{(x^3+x)'}|_{x=0} = 1$.

The pole on upper half plane is $x = i$, it is also a simple pole, so $Res(f(z)e^{iz}, i) = \frac{e^{ix}}{(x^3+x)'}|_{x=i} = -\frac{e^{-1}}{2}$.

So by the formula $P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)} dx = 2\pi i Res(f(z)e^{iz}, i) + \pi i Res(f(z)e^{iz}, 0) = 2\pi i \cdot (-\frac{e^{-1}}{2}) + \pi i \cdot 1 = (1 - e^{-1})\pi i$. So $P.V. \int_{-\infty}^{\infty} \frac{\sin x}{x(x^2+1)} dx = Im(P.V. \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)} dx) = (1 - e^{-1})\pi$ ☐